Generative modelling of multivariate geometric extremes using normalising flows

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Joint work with Raphaël Huser[2], Ioannis Papastathopoulos[1], Jordan Richards[1]

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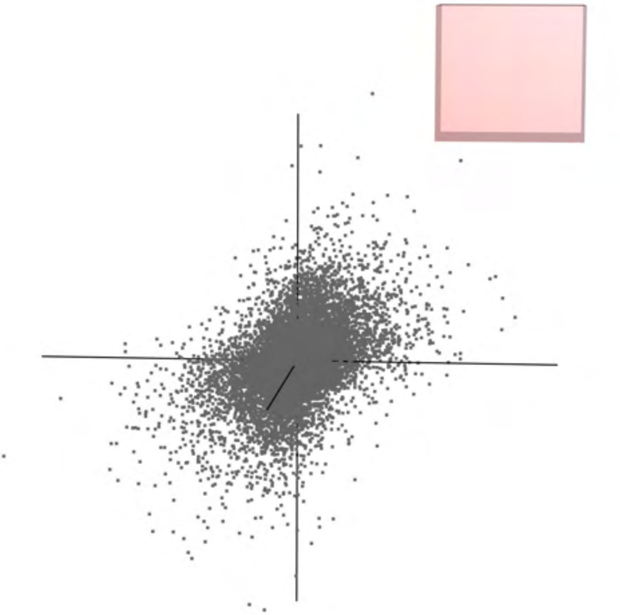
Very broad overview



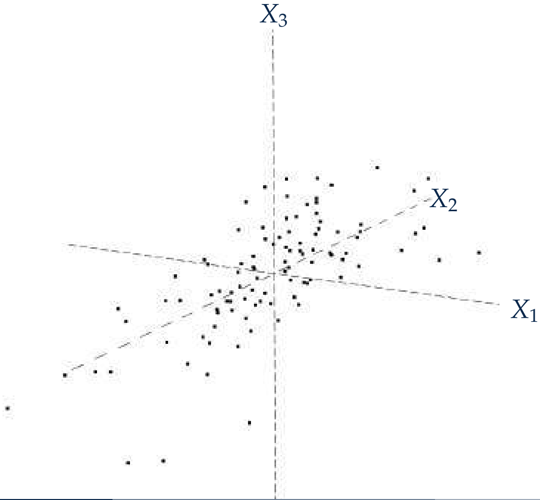
Presentation Overview

1. Geometric extreme value theory
2. Statistical inference
3. Simulation study
4. An application to low and high wind speeds

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots \in \mathbb{R}^d$ be iid draws from $\mathbb{P}_{\boldsymbol{X}}$ with standard Laplace marginal distributions, and

$$N_n := \left\{ \frac{\boldsymbol{X}_1}{\log(n)}, \ldots, \frac{\boldsymbol{X}_n}{\log(n)} \right\}$$

$n = 100$



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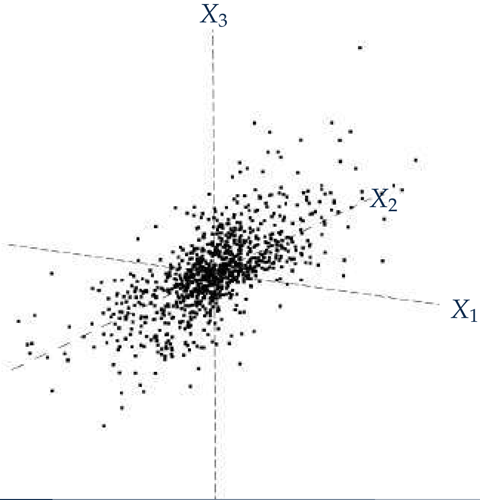

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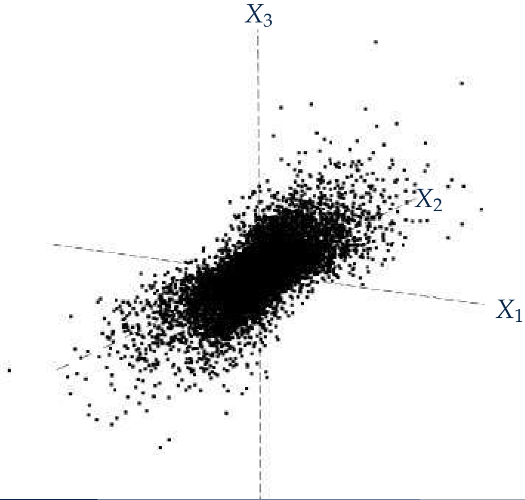

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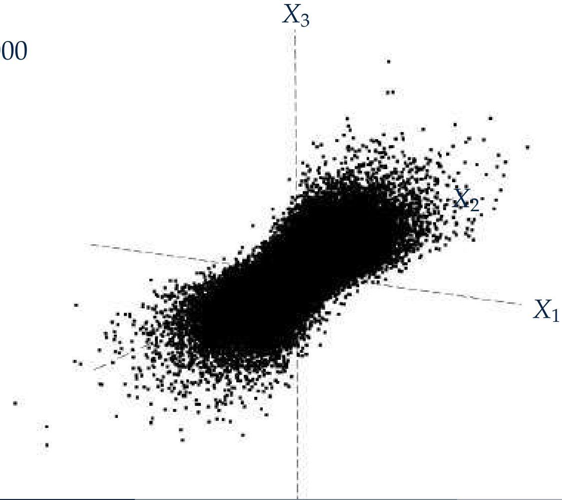

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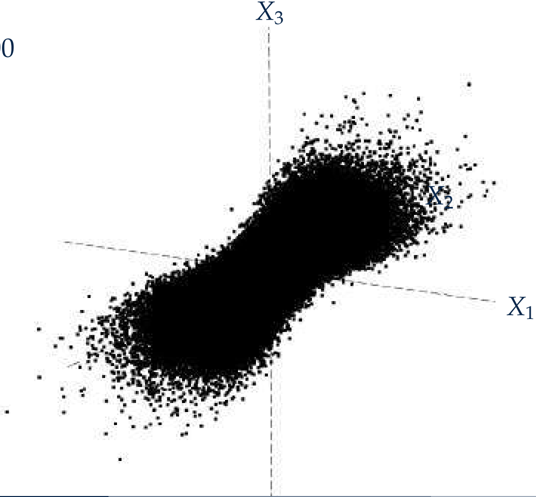

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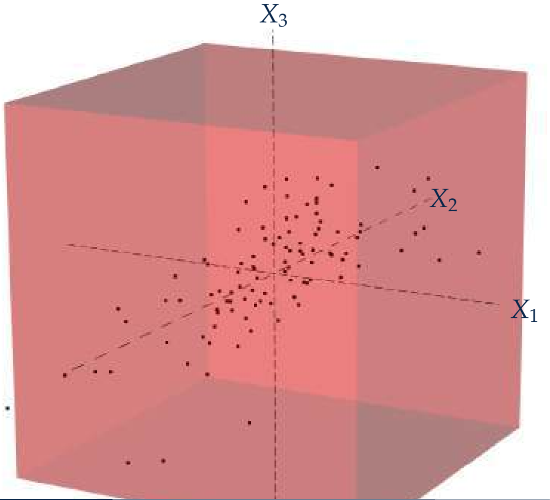

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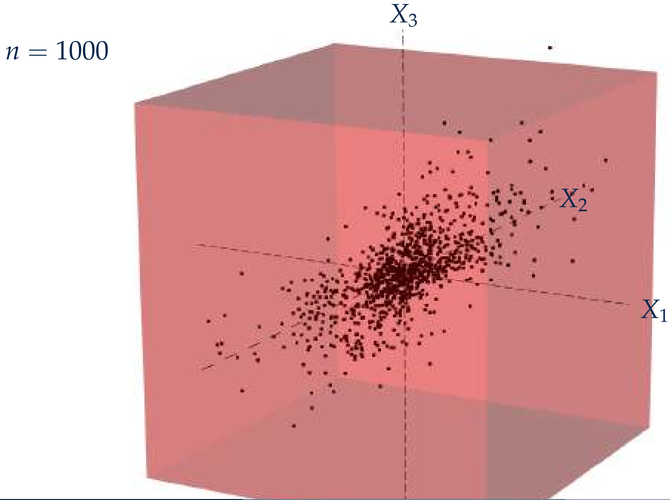

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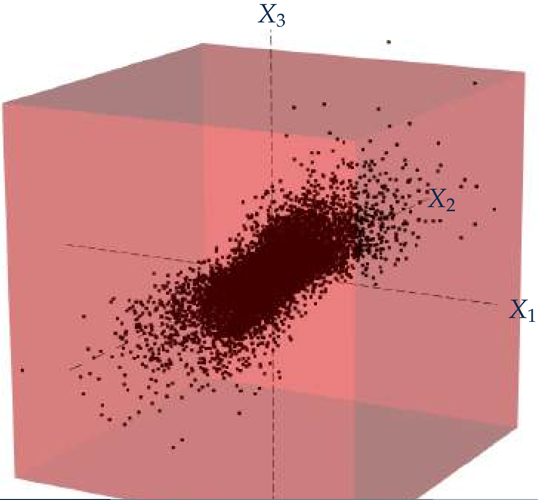

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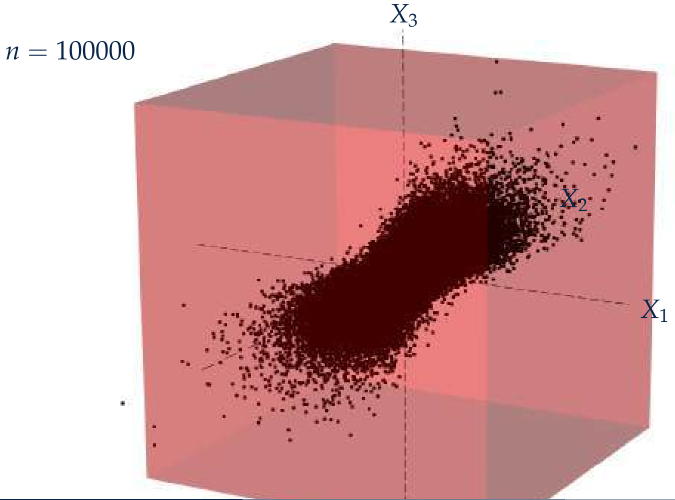

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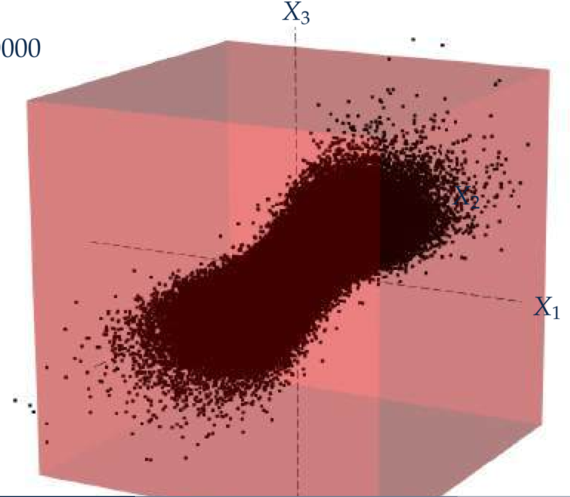

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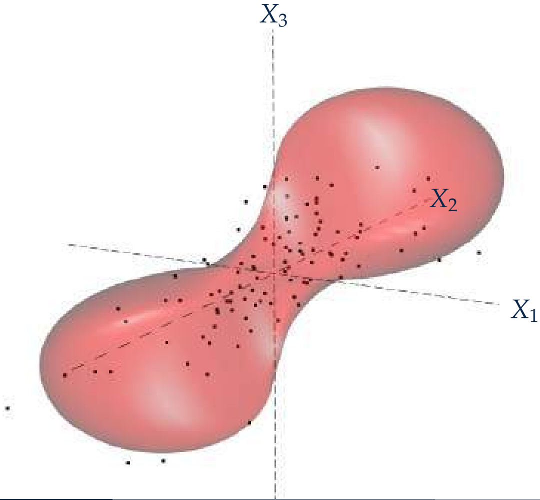

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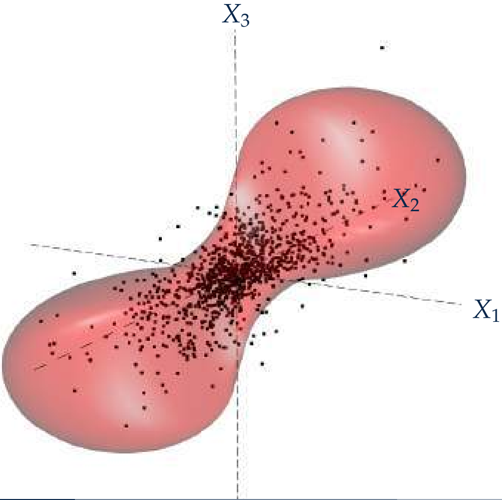

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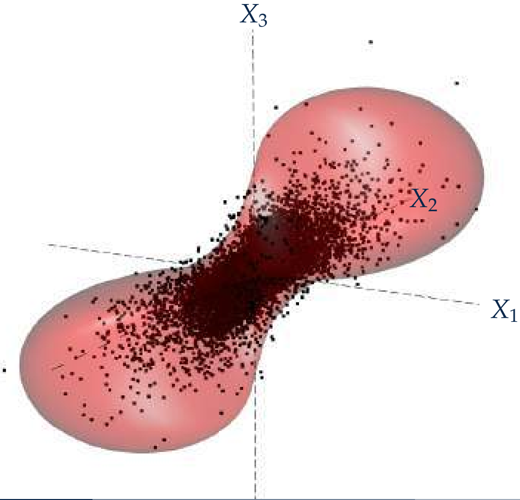

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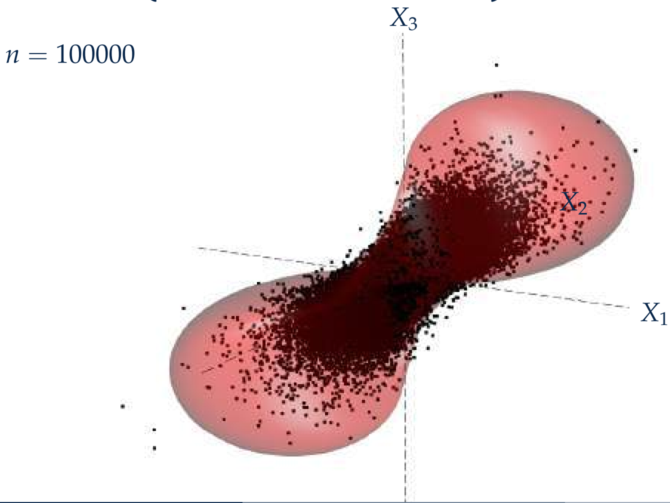

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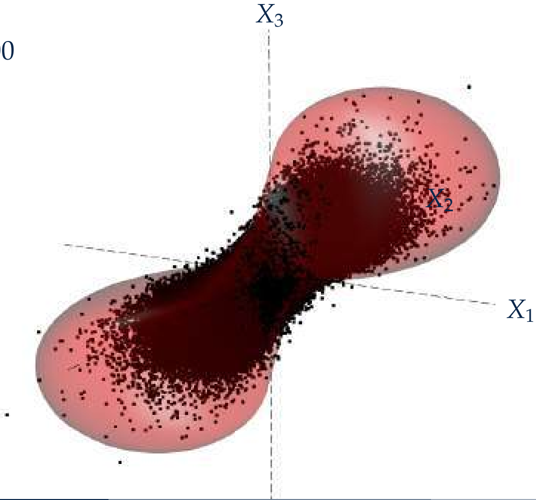

Very broad overview – A directional statistics approach

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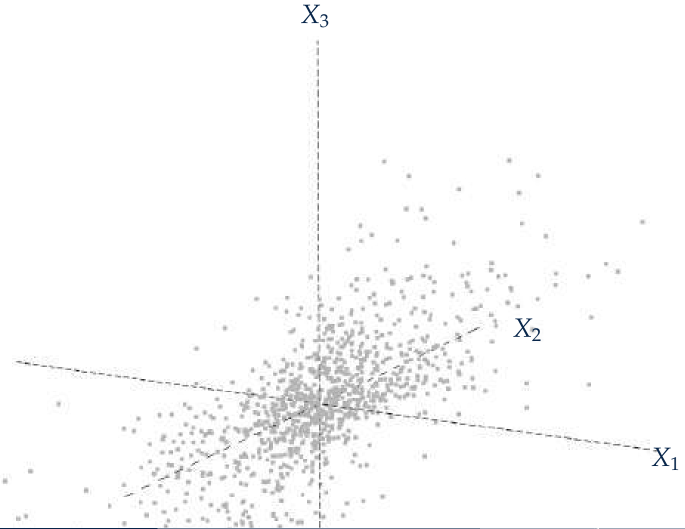

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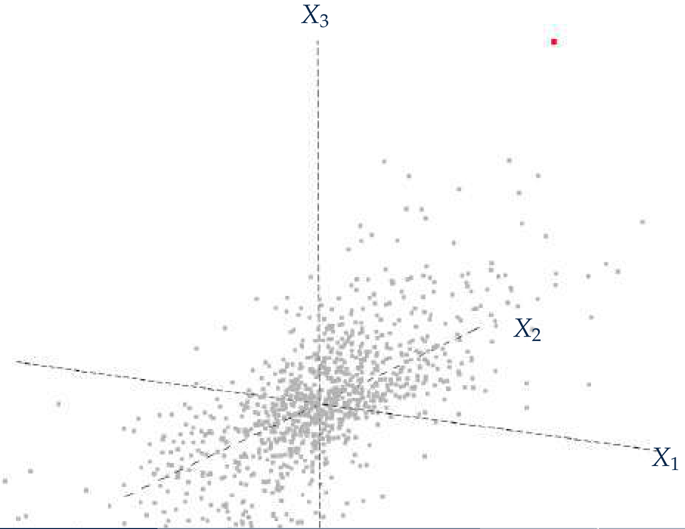

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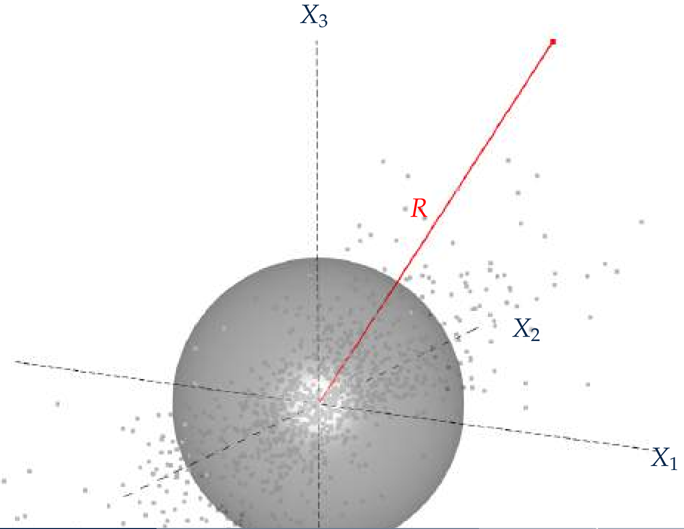
$$R := \|\boldsymbol{X}\|_2 > 0, \qquad \boldsymbol{W} := \boldsymbol{X}/R \in \mathbb{S}^{d-1}.$$



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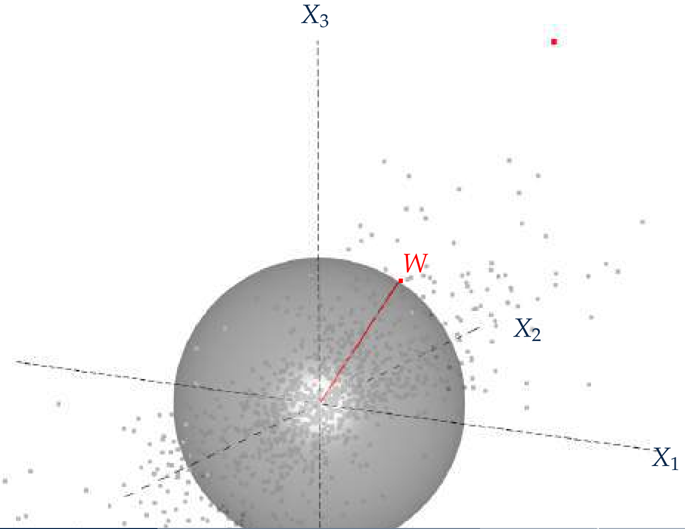


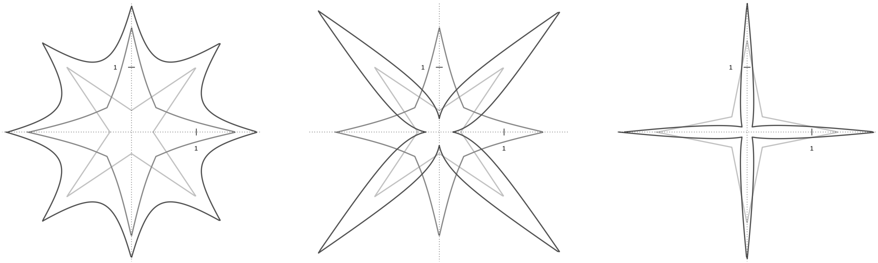
Starshaped sets ★ – A basis for our model construction

- A set $\mathcal{B} \in \mathbb{R}^d$ is starshaped if there exists a set $\ker(\mathcal{B}) \subseteq \mathcal{B}$ such that for $\boldsymbol{x} \in \ker(\mathcal{B})$ and for all $\boldsymbol{y} \in \mathcal{B}$, the segment $[\boldsymbol{x} : \boldsymbol{y}] \in \mathcal{B}$.

- A set $\mathcal{B} \in \star$ is in one-to-one correspondence with a radial function

$$r_{\mathcal{B}}(\boldsymbol{w}) = \sup\{\lambda \in \mathbb{R} \,:\, \lambda \boldsymbol{w} \in \mathcal{B}\}, \quad \boldsymbol{w} \in \mathbb{S}^{d-1}.$$

- Starshaped sets admit algebraic operations via their radial functions:



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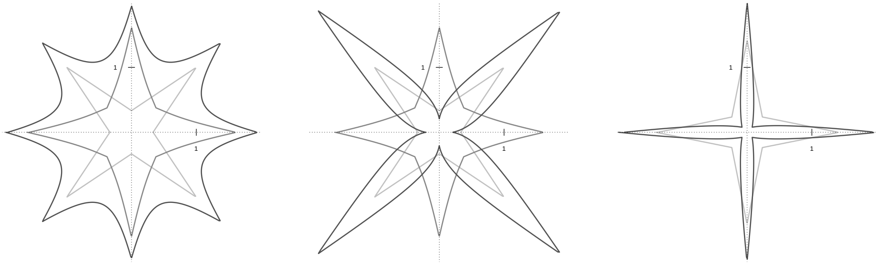

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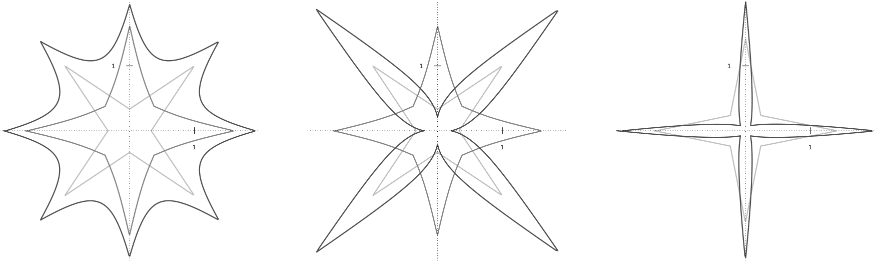

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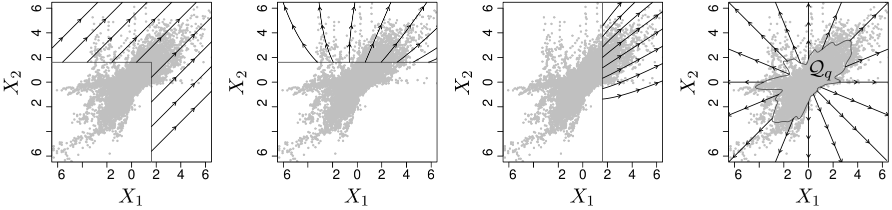
Geometric multivariate EVT: Motivation

- Interest in gaining more insight into the (extremal) dependence structure of a random vector $\boldsymbol{X} = (X_1, \ldots, X_d) \in \mathbb{R}^d$.

- Extrapolating beyond the range of observed data:



Directions along which MEVT frameworks allow extrapolation to tail regions: (a) MRV, (b) and (c) conditional extremes, (d) geometric extremes.

- Effectively, we want to study the random variable

$$(R, \boldsymbol{W}) \mid \{R \notin \mathcal{Q}_q\}.$$

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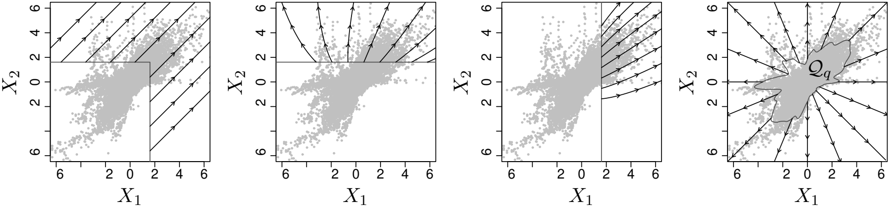

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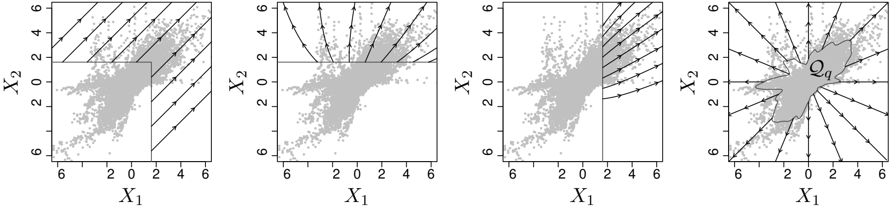

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The quantile set $\mathcal{Q}_q$

- We let $\mathcal{Q}_q$ via the q-th quantile of $R \mid \boldsymbol{W} = \boldsymbol{w}$, that is, it satisfies

$$\mathbb{P}[R \leq r_{\mathcal{Q}_q}(\boldsymbol{w}) \mid W = \boldsymbol{w}] = q, \quad \text{for all } \boldsymbol{w} \in \mathbb{S}^{d-1}.$$

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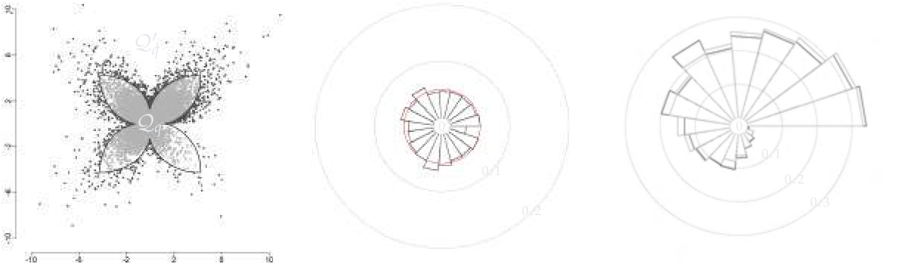
$$\mathbb{P}[\boldsymbol{X} \notin \mathcal{Q}_q] = 1 - q, \qquad \text{and} \qquad \boldsymbol{W} \mid \{\boldsymbol{X} \notin \mathcal{Q}_q\} \stackrel{d}{=} \boldsymbol{W}.$$



Left: Independent samples ($n = 2 \times 10^4$) from a bivariate distribution having true quantile set $\mathcal{Q}_{0.95}$, boundary $\partial\mathcal{Q}_{0.95}$ (solid black line) and complement $\mathcal{Q}'_{0.95}$. Centre: Empirical proportion of exceedances binned by angular regions with true exceedance probability (0.05) in red. Right: Circular histogram of the density of all sampled angles (light grey) and of exceedance angles (dark grey) with concentric circles denoting density level sets.

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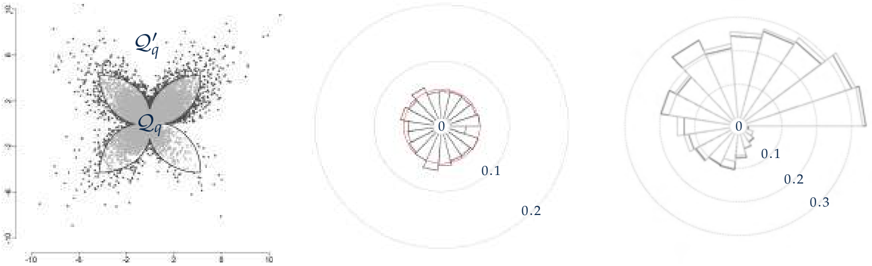

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Exceedances of $\mathcal{Q}_q$

- Note that the event $\{\boldsymbol{X} = R\boldsymbol{W} \notin \mathcal{Q}_q\}$ corresponds to $\{R > r_{\mathcal{Q}_q}(\boldsymbol{W})\}$.

- Further, Papastathopoulos et al. (2023) show conditions under which there exist a starshaped set G such that

$$\left(\frac{R - r_{\mathcal{Q}_q}(\boldsymbol{W})}{r_{\mathcal{G}}(\boldsymbol{W})}, \boldsymbol{W}\right) \mid \{R > r_{\mathcal{Q}_q}(\boldsymbol{W})\} \xrightarrow{d} (Z, \boldsymbol{V}), \quad \text{as } q \to 1, \tag{1}$$

where $Z \sim \mathrm{Exp}(1)$, $\boldsymbol{V} \sim \mathbb{P}_{\boldsymbol{W}}$.

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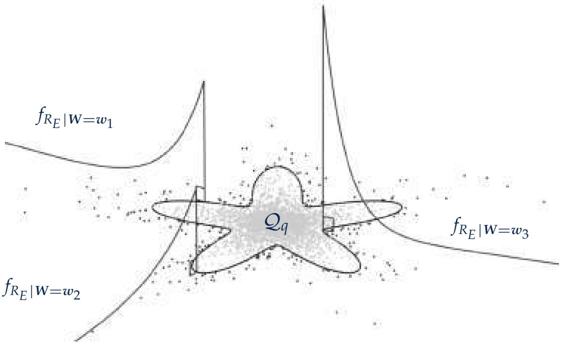

PROPOSED MODELS

Links between parameters and models

Under appropriate convergence conditions[1], it can be shown that the quantile set $\mathcal{Q}_q$ is asymptotically ($q \to 1$) a scale multiple of the scaling/limit set $\mathcal{G}$, that is,

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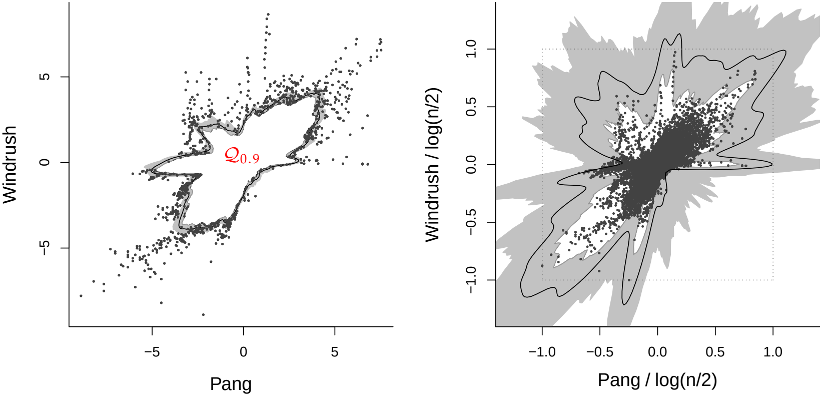

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$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_0(r_{\mathcal{G}}^{-1}(\boldsymbol{x})), \quad x \in \mathbb{R}^d,$$

then $\mathcal{G}$ and $\mathcal{W}$ can be linked[1] through

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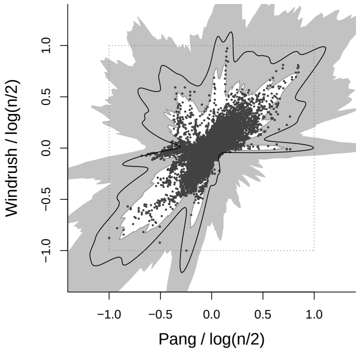

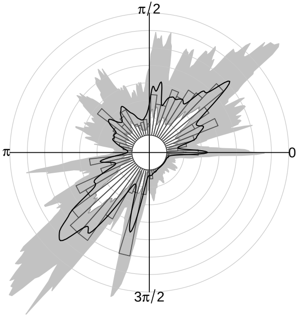

[1] Papastathopoulos et al. (2023)

Proposed models

- Any positive function $r_{\mathcal{B}}$ defined on $\mathbb{S}^{d-1}$ can be written as

$$r_{\mathcal{B}}(\boldsymbol{w}) = \beta_{\mathcal{B}} f_{\mathcal{B}}(\boldsymbol{w}), \quad \boldsymbol{w} \in \mathbb{S}^{d-1},$$

for a constant $\beta_{\mathcal{B}} = \int_{\mathbb{S}^{d-1}} r_{\mathcal{B}}(\boldsymbol{w})\, \mathrm{d}\boldsymbol{w}$ and density $f_{\mathcal{B}}$ integrating to 1 on $\mathbb{S}^{d-1}$.

- Using the links $\mathcal{G}$–$\mathcal{Q}_q$ and $\mathcal{G}$–$\mathcal{W}$, we can formulate a statistical model

$$r_{\mathcal{Q}_q}(\boldsymbol{w}) = \beta_{\mathcal{Q}_q} f_W(\boldsymbol{w})^d \text{ and } r_{\mathcal{G}}(\boldsymbol{w}) = \beta_{\mathcal{G}} f_W(\boldsymbol{w})^d, \qquad \boldsymbol{w} \in \mathbb{S}^{d-1}.$$

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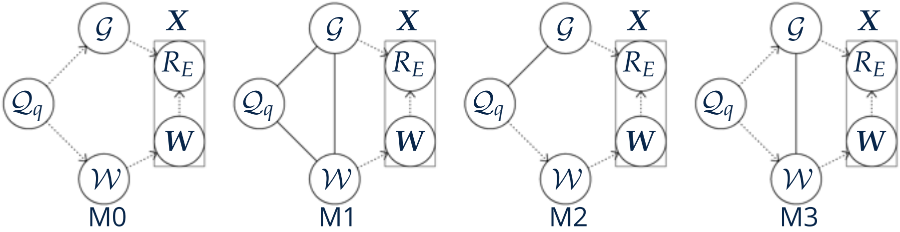

Proposed models

- We introduce a deformation set $\mathcal{D}$ with radial function $f_{\mathcal{D}} : \mathbb{S}^{d-1} \to [0, \infty)$.

- We can then weaken the equality assumptions of models M1, M2, and M3 via

$$r_{\mathcal{Q}_q}(\boldsymbol{w}) = \beta_q f_{\mathcal{D}}(\boldsymbol{w}) f_{\mathcal{G}}(\boldsymbol{w}), \quad \boldsymbol{w} \in \mathbb{S}^{d-1},$$

and

$$r_{\mathcal{G}}(\boldsymbol{w}) = \beta_{\mathcal{G}} \{f_{\mathcal{D}}(\boldsymbol{w}) f_{W}(\boldsymbol{w})\}^{1/d}, \quad \boldsymbol{w} \in \mathbb{S}^{d-1},$$

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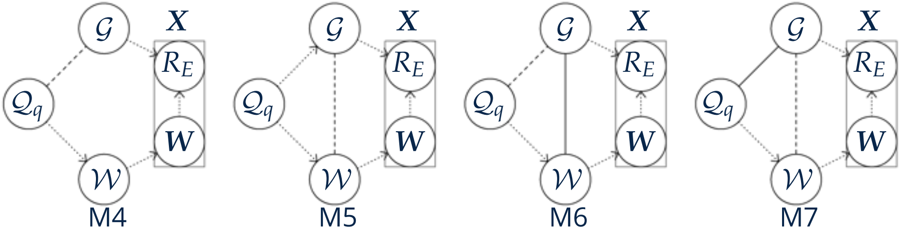

- Models M4 to M7 are identifiable as is, but we impose penalisation on $\mathcal{D}$.

STATISTICAL INFERENCE

Normalising flows[1] and density estimation[2]

- A normalising flow (NF) learns a transformation mapping a random variable $Y \in \mathcal{Y}$ with unknown distribution to that of a known, base variable $Z \in \mathcal{Z}$.

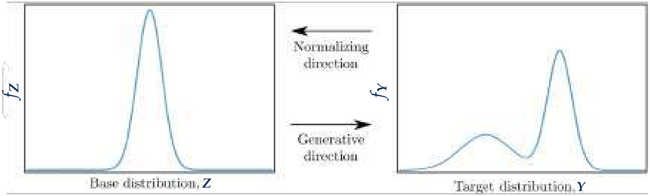

Figure 1 of Kobyzev et al. (2021)

[1]Tabak & Vanden-Eijnden (2010), [2]Dinh et al. (2015)

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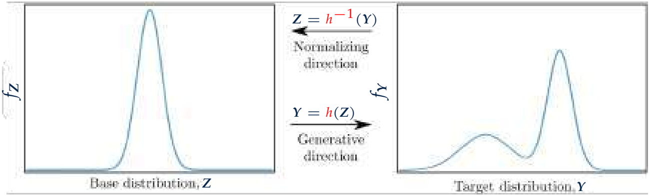


Figure 1 of Kobyzev et al. (2021)

- Assuming Y admits a density on $\mathcal{Y}$, this problem can be phrased as aiming to infer a (bijective and differentiable) transformation function h such that

$$f_Y(\boldsymbol{y}) = f_Z\{h^{-1}(\boldsymbol{y})\} \left| \frac{\partial h^{-1}(\boldsymbol{y})}{\partial \boldsymbol{y}} \right|, \quad \boldsymbol{y} \in \mathcal{Y}.$$

In practice, h is modelled as a composition of many simple bijective transformations $h_1, \ldots, h_k$, *i.e.* $h = h_1 \circ h_2 \circ \ldots \circ h_k$.

[1]Tabak & Vanden-Eijnden (2010), [2]Dinh et al. (2017)

A map from the hypersphere to the hypercylinder

- Transform the observations and models from $\mathbb{S}^{d-1}$ to a cylindrical space $\mathbb{C}^{d-1}$ (by abuse of notation).

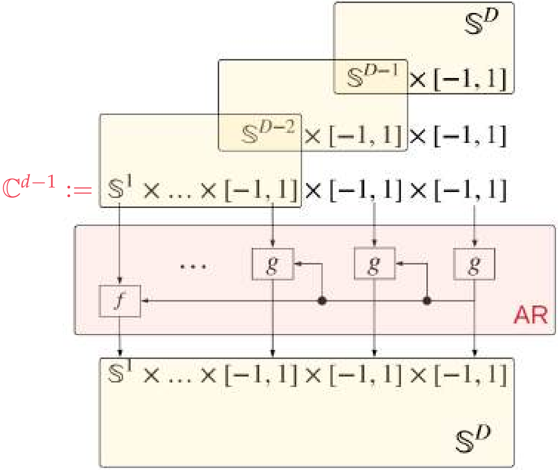
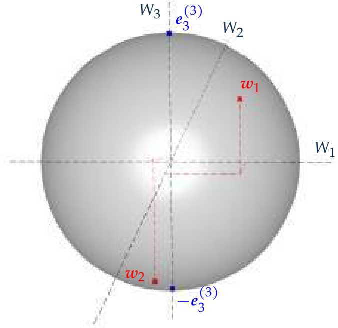


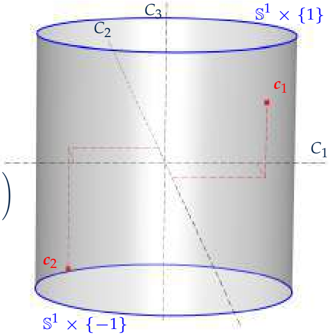
Figure 6 of Rezende et al. (2020)

A map from the hypersphere $\mathbb{S}^2$ to the hypercylinder $\mathbb{C}^2$



$$T : \mathbb{S}^2 \backslash \mathcal{S}_3 \to \mathbb{C}^2$$

$$w \mapsto c = \left(\frac{w_1}{\sqrt{1-w_3^2}}, \frac{w_2}{\sqrt{1-w_3^2}}, w_3 \right)$$



A model for PDFs and positive functions on $\mathbb{S}^{d-1}$

- It follows from the map T that a target PDF $f_{\mathcal{B}} : \mathbb{S}^{d-1} \backslash \mathcal{S}_d \to [0, \infty)$, describing the shape of a starshaped set $\mathcal{B} \in \mathbb{R}^d$ *a.e.*, can be written as

$$f_{\mathcal{B}}(\boldsymbol{w}) = f_Y(T(\boldsymbol{w}))|\partial T(\boldsymbol{w})/\partial \boldsymbol{w}|, \quad \boldsymbol{w} \in \mathbb{S}^{d-1} \backslash \mathcal{S}_d,$$

for a target PDF f_Y defined on $\mathbb{C}^{d-1}$.

- Using the NFs formulation, $f_{\mathcal{B}}$ can in turn be modelled in terms of a known base PDF $f_Z : \mathbb{C}^{d-1} \to [0, \infty)$ and a normalising flow $h_{\mathcal{B}}$ as

$$f_{\mathcal{B}}(\boldsymbol{w}) = f_Z\{h_{\mathcal{B}}^{-1}(T(\boldsymbol{w}))\} \left| \frac{\partial h_{\mathcal{B}}^{-1}(T(\boldsymbol{w}))}{\partial T(\boldsymbol{w})} \right| \left| \frac{\partial T(\boldsymbol{w})}{\partial \boldsymbol{w}} \right|, \quad \boldsymbol{w} \in \mathbb{S}^{d-1} \backslash \mathcal{S}_d,$$

where $|\partial T(\boldsymbol{w})/\boldsymbol{w}|$ is the Jacobian of the recursive transformation T.

- Further, a model for any positive/radial function $r_{\mathcal{B}}$ of a starshaped set $\mathcal{B}$ – such as the quantile set $\mathcal{Q}_q$ or the scaling set $\mathcal{G}$ – can be obtained via

$$r_{\mathcal{B}} = \beta_{\mathcal{B}} f_{\mathcal{B}}$$

where $f_{\mathcal{B}}$ is as above, and $\beta_{\mathcal{B}} > 0$ is a coefficient to be learned alongside the NF $h_{\mathcal{B}}$.

[1] Stimper et al. (2023)

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A GRADIENT DESCENT APPROACH

A PyTorch[1] implementation[2] of NFs and composite loss minimisation via the Adam optimiser[3]

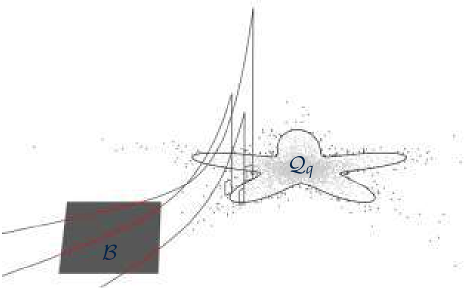
[1] Paszke et al. (2019), [2] Stimper et al. (2023), [3] Kingma & Ba (2017)

PROBABILITY ESTIMATION

Probability estimation

- For any Borel set $\mathcal{B} \in \mathbb{R}^d \backslash \mathcal{Q}_q$,

$$\mathbb{P}[\boldsymbol{X} \in \mathcal{B} \mid \boldsymbol{X} \notin \mathcal{Q}_q] = \int_{\mathbb{S}^{d-1}} \int_{\mathcal{B} \cap]\mathbf{0}:\boldsymbol{w})} \frac{1}{r_{\mathcal{G}}(\boldsymbol{w})} \exp\left\{-\frac{r - r_{\mathcal{Q}_q}(\boldsymbol{w})}{r_{\mathcal{G}}(\boldsymbol{w})}\right\} f_{\boldsymbol{W}}(\boldsymbol{w}) \mathrm{d}r \, \mathrm{d}\boldsymbol{w}.$$

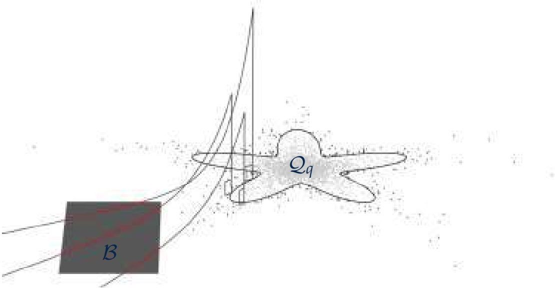


Probability estimation

- For any Borel set $\mathcal{B} \in \mathbb{R}^d \setminus \mathcal{Q}_q$, we use the Monte Carlo integration

$$\mathbb{P}[\boldsymbol{X} \in \mathcal{B} \mid \boldsymbol{X} \notin \mathcal{Q}_q] \overset{\mathbb{P}}{\leftarrow} \frac{1}{m} \sum_{i=1}^{m} \int_{\mathcal{B} \cap]\mathbf{0}:\boldsymbol{w}_i)} \frac{1}{r_{\mathcal{G}}(\boldsymbol{w}_i)} \exp\left\{ -\frac{r - r_{\mathcal{Q}_q}(\boldsymbol{w}_i)}{r_{\mathcal{G}}(\boldsymbol{w}_i)} \right\} \mathrm{d}r, \quad n \to \infty.$$

where $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \sim f_{\boldsymbol{W}}$.



- The integral is exact provided one knows all radial entry and exit points of $\mathcal{B}$.

- The collection $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \sim f_{\boldsymbol{W}}$ is sampled fast using the generative direction of the NF.

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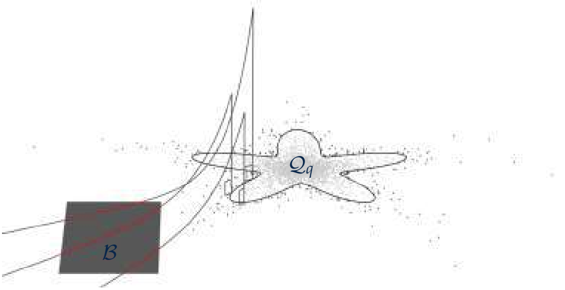

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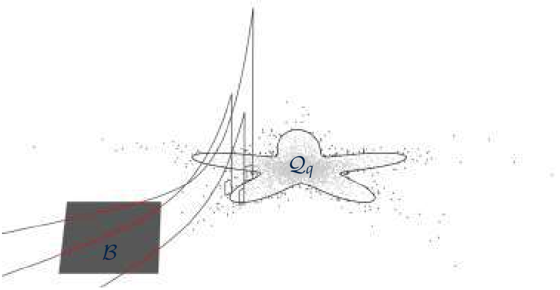

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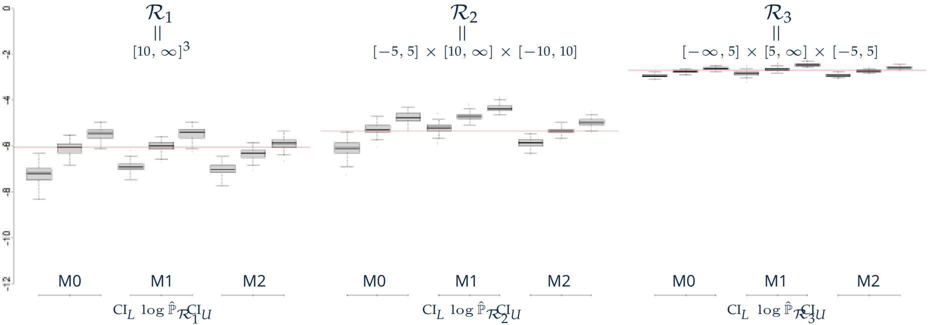
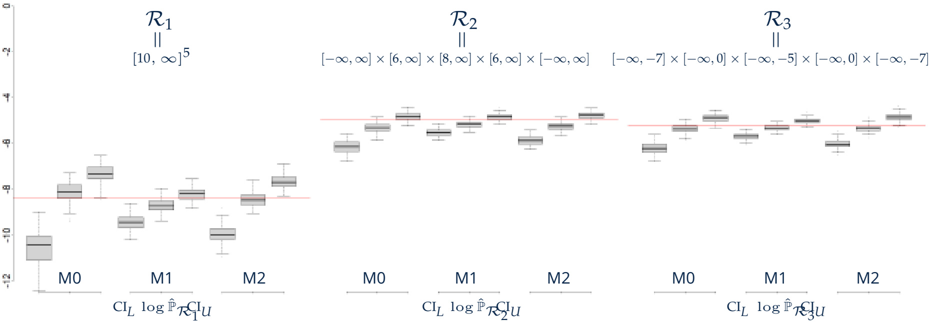
Simulation study results – 3 dimensions



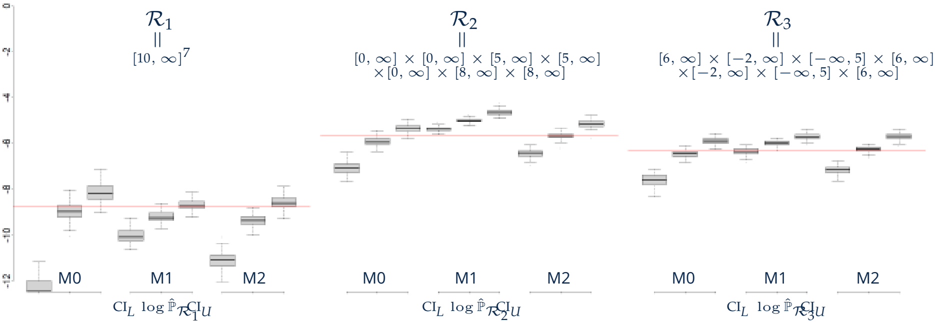
Boxplots of 100 estimated log-probabilities and associated lower- and upper-bounds of 95% bootstrap confidence intervals for the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathbb{R}^3$. ($n = 10^4$).

Simulation study results – 5 dimensions

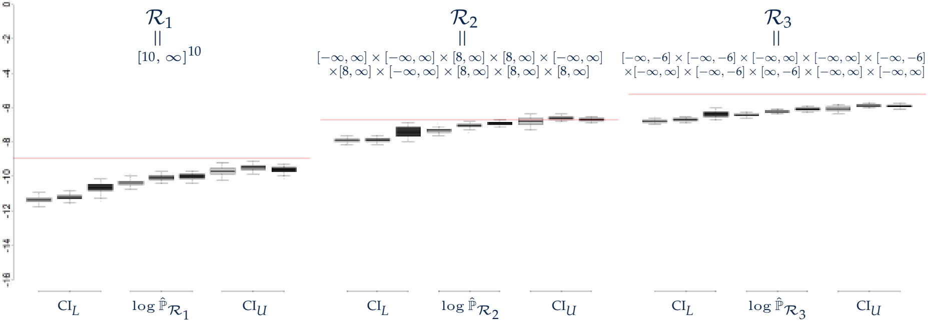


Boxplots of 100 estimated log-probabilities and associated lower- and upper-bounds of 95% bootstrap confidence intervals for the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathbb{R}^5$. ($n = 10^4$).

Simulation study results – 7 dimensions



Boxplots of 100 estimated log-probabilities and associated lower- and upper-bounds of 95% bootstrap confidence intervals for the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathbb{R}^7$. ($n = 10^4$).



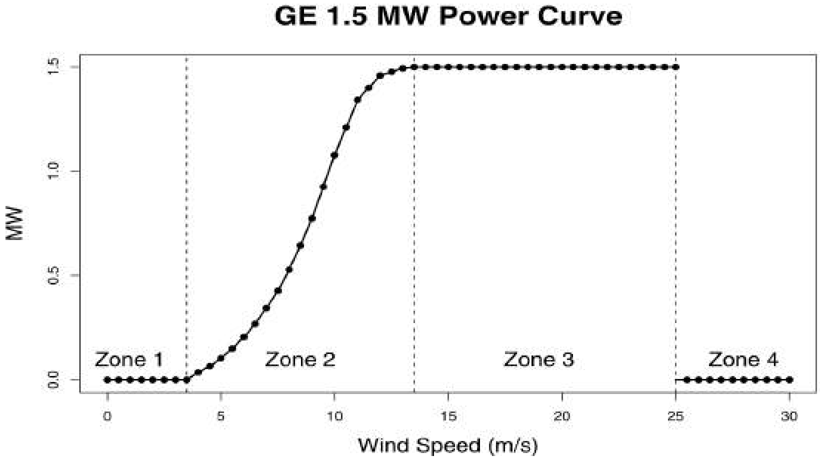
Boxplots of 100 estimated log-probabilities and associated lower- and upper-bounds of 95% bootstrap confidence intervals for the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathbb{R}^{10}$. ($n = 5 \times 10^4$, 10^5, 2×10^5).

LOW AND HIGH WIND SPEEDS

In relation to electricity production in the Pacific Northwest, United States

Pacific Northwest region of the United States

Wind turbine power curves



Scale-shape homogenisation

- Define the windspeed

$$X^{o}_{j,m,h}$$

at station j in month m of the year and hour h.

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at station j in month m of the year and hour h.

- We assume[1]

$$X^{\text{o}}_{j,m,h} \sim \text{Weibull}\big(\lambda_{j,m,h} = s_{j,1}(m) + s_{j,2}(h)\,,\ \kappa_{j,m,h} = s_{j,3}(m) + s_{j,4}(h)\big), \tag{3}$$

where s denotes a cubic cyclic spline on $m \in \{1, \ldots, 12\}$ or $h \in \{0, \ldots, 23\}$.

[1] Elliott et al. (2004)

Scale-shape homogenisation

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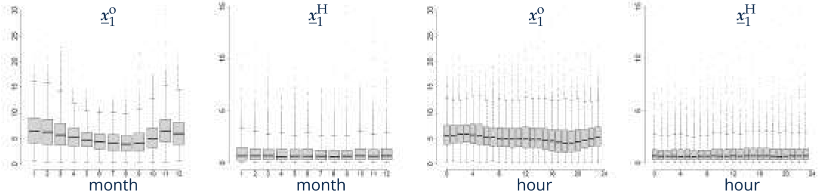
where s denotes a cubic cyclic spline on $m \in \{1, \ldots, 12\}$ or $h \in \{0, \ldots, 23\}$.

- We fit the model using `evgam`[2] and apply $X^{\mathrm{H}}_{j,m,h} := (X^{\mathrm{o}}_{j,m,h}/\hat{\lambda}_{j,m,h})^{\hat{\kappa}_{j,m,h}}$



[1] Elliott et al. (2004), [2] Youngman (2022)

Analysis of station configurations – January at 18:00

(a) Minimises probability of no production

Analysis of station configurations – January at 18:00

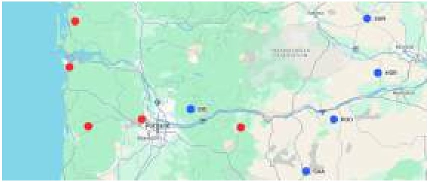
(a) Minimises probability of no production

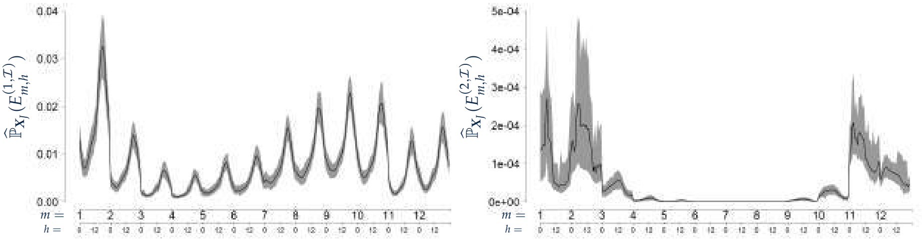
(b) Maximises probability of no production

Analysis of station configurations – January at 18:00


(a) Minimises probability of no production


(b) Maximises probability of no production


(c) Maximises probability of full production



Month m, Hour h Month m, Hour h

Configuration (a): Minimises probability of no production

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Proposed models

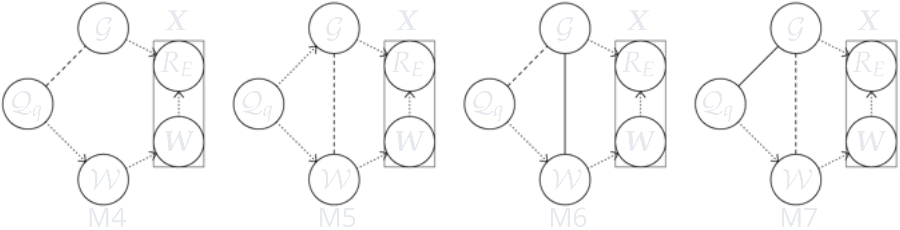
- We introduce a deformation set $\mathcal{D}$ with radial function $f_{\mathcal{D}} : \mathbb{S}^{d-1} \to [0, \infty)$.

- We can then weaken the equality assumptions of models M1, M2, and M3 via

$$r_{\mathcal{Q}_q}(w) = \beta_q f_{\mathcal{D}}(w) f_{\mathcal{G}}(w), \quad w \in \mathbb{S}^{d-1},$$

and

$$r_{\mathcal{G}}(w) = \beta_{\mathcal{G}} \{f_{\mathcal{D}}(w) f_W(w)\}^{1/d}, \quad w \in \mathbb{S}^{d-1},$$



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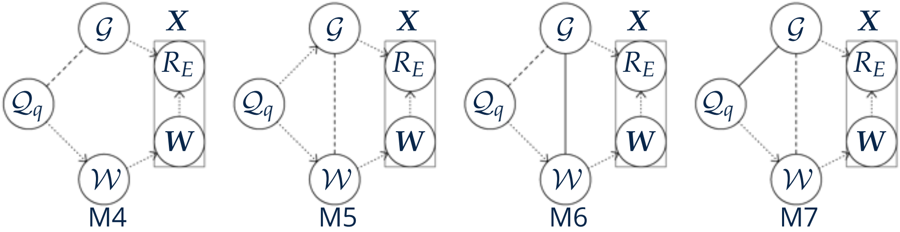

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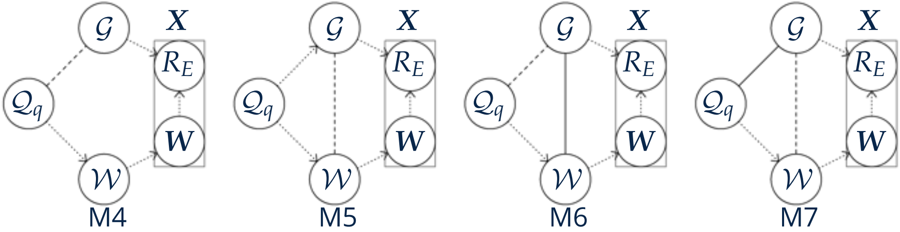

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Model fitting via loss minimisation

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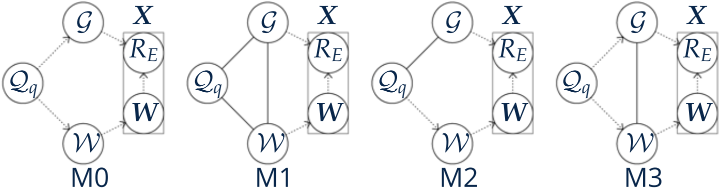

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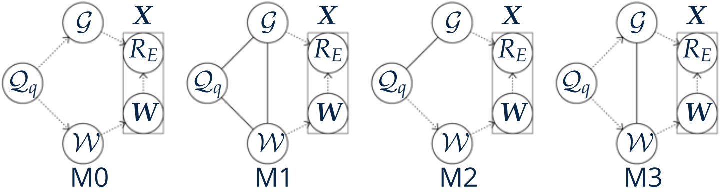

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  - for $\mathcal{Q}_q$:

$$\mathcal{L}_{\mathcal{Q}_q}(\beta_{\mathcal{Q}_q}, f_{\mathcal{Q}_q}\,;\, \underline{x}) = \frac{1}{n}\sum_{i=1}^{n} \max\left\{(1-q)\left[\|x_i\| - \beta_{\mathcal{Q}_q} f_{\mathcal{Q}_q}\left(\frac{x_i}{\|x_i\|}\right)\right], q\left[\|x_i\| - \beta_{\mathcal{Q}_q} f_{\mathcal{Q}_q}\left(\frac{x_i}{\|x_i\|}\right)\right]\right\}.$$

  - for $\mathcal{G}$:

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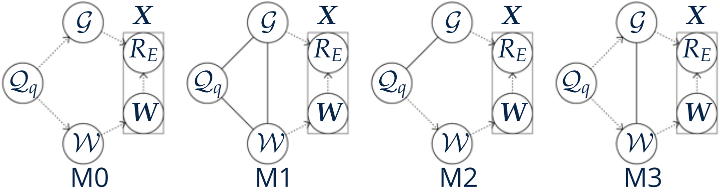

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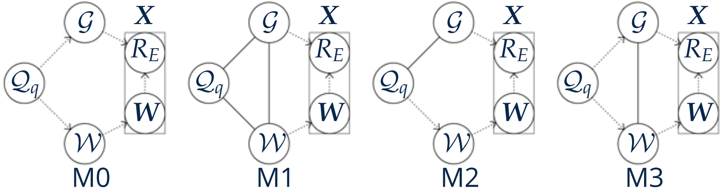

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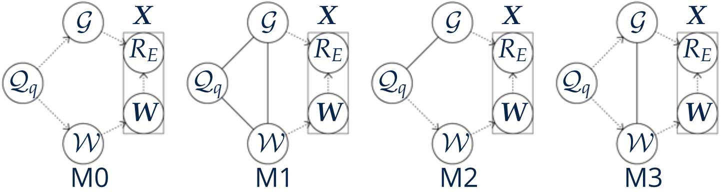

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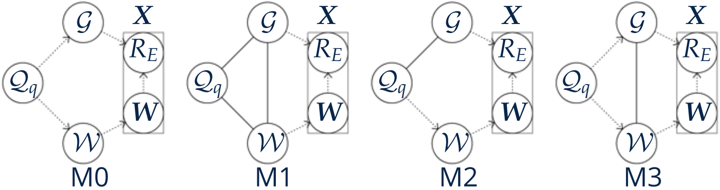

- Model M1 is fitted by sequentially minimising the loss $\mathcal{L}_{\mathcal{Q}_q,\mathcal{G},\mathcal{W}}(\beta_{\mathcal{Q}_q}, \beta_{\mathcal{G}}, f_W\,;\,\underline{\boldsymbol{x}}) =$

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- Comments on M2 and M3.

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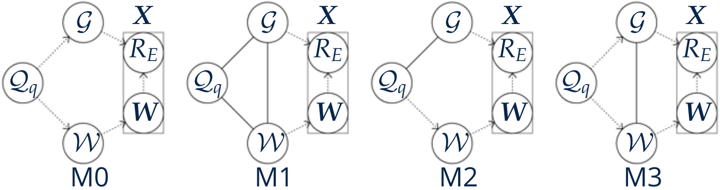

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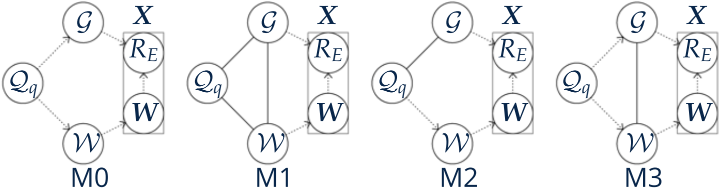

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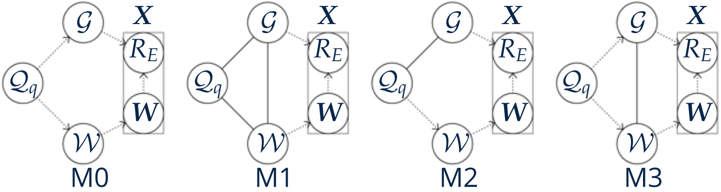

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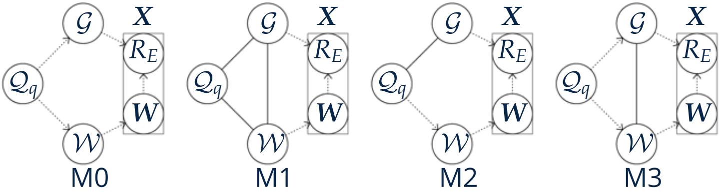

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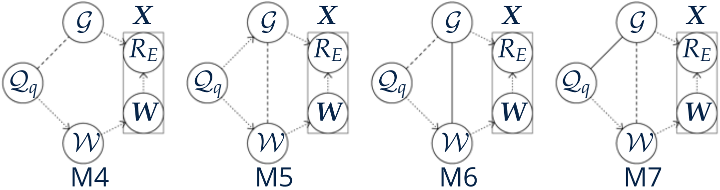

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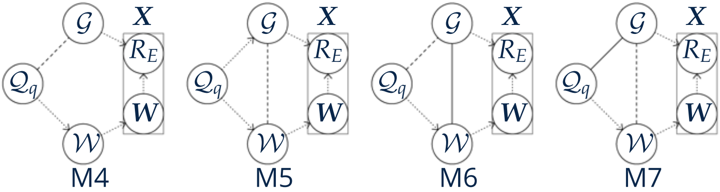

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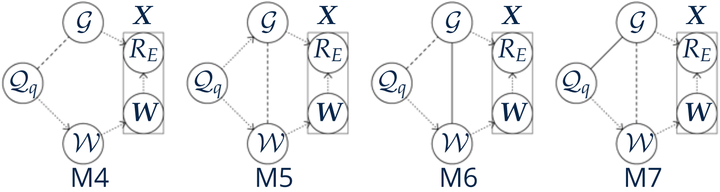

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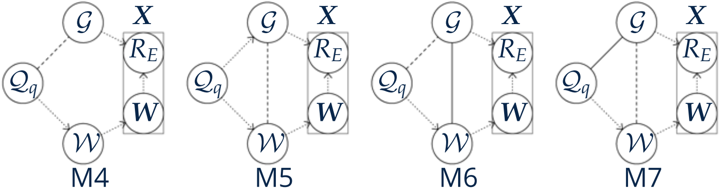

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Uniform-on-$\mathbb{S}^{d-1}$ penalisation

- To devise the uniform density on $\mathbb{S}^{d-1}$, we consider $A_{d-1}(r)$ the hypervolume (or surface area) of the $(d-1)$-sphere of radius r given by

$$A_{d-1}(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}, \quad r \in (0, \infty),$$

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- It follows that a PDF with uniform density on $\mathbb{S}^{d-1}$ is given by

$$f_U(w) = 1/A_{d-1}(1)$$

  for all $w \in \mathbb{S}^{d-1}$.

- Penalisation of $f_{\mathcal{D}}$ away from f_U can then be performed via the Kullback–Leibler divergence $\mathrm{D}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}] = \int_{\mathbb{S}^{d-1}} \log[f_U(w)/f_{\mathcal{D}}(w)] f_U(w)\, dw$.

- In practice, this integral is approximated via Monte Carlo integration by sampling a large number m of directions $u_1, \ldots, u_m$ uniformly on $\mathbb{S}^{d-1}$ and calculating

$$\overline{\mathrm{D}}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}] := \frac{1}{m} \sum_{i=1}^{m} \log[f_U(u_i)/f_{\mathcal{D}}(u_i)] = -\log[A_{d-1}(1)] - \frac{1}{m} \sum_{i=1}^{m} \log[f_{\mathcal{D}}(u_i)], \qquad (5)$$

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- Penalisation of $f_{\mathcal{D}}$ away from f_U can then be performed via the Kullback–Leibler divergence $\mathrm{D}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}] = \int_{\mathbb{S}^{d-1}} \log[f_U(\boldsymbol{w})/f_{\mathcal{D}}(\boldsymbol{w})] f_U(\boldsymbol{w})\, d\boldsymbol{w}$.

- In practice, this integral is approximated via Monte Carlo integration by sampling a large number m of directions $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ uniformly on $\mathbb{S}^{d-1}$ and calculating

$$\overline{\mathrm{D}}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}] := \frac{1}{m} \sum_{i=1}^{m} \log[f_U(\boldsymbol{u}_i)/f_{\mathcal{D}}(\boldsymbol{u}_i)] = -\log[A_{d-1}(1)] - \frac{1}{m} \sum_{i=1}^{m} \log[f_{\mathcal{D}}(\boldsymbol{u}_i)], \qquad (5)$$

  with $\overline{\mathrm{D}}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}] \xrightarrow{\mathbb{P}} \mathrm{D}_{\mathrm{KL}}[f_U \| f_{\mathcal{D}}]$ as $m \to \infty$.

Uniform-on-$\mathbb{S}^{d-1}$ penalisation

- To devise the uniform density on $\mathbb{S}^{d-1}$, we consider $A_{d-1}(r)$ the hypervolume (or surface area) of the $(d-1)$-sphere of radius r given by

$$A_{d-1}(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}, \quad r \in (0, \infty),$$

  where Γ denotes the gamma function.

- It follows that a PDF with uniform density on $\mathbb{S}^{d-1}$ is given by

$$f_U(\boldsymbol{w}) = 1/A_{d-1}(1)$$

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Model assessment

Model assessment – A random point measure approach

- Under assumptions of uniform convergence on $\mathbb{S}^{d-1}$,

$$F_{R|\boldsymbol{W}}\left(\frac{R - r_{\mathcal{Q}_q^\star}(\boldsymbol{W})}{r_{\mathcal{G}^\star}(\boldsymbol{W})}\right)^{1/d} \boldsymbol{W}\Big| \left\{R > r_{\mathcal{Q}_q^\star}(\boldsymbol{W})\right\} \xrightarrow{d} \boldsymbol{U}_{B_1(\boldsymbol{0})}, \quad \text{as } q \to 1,$$

  where $r_{\mathcal{Q}_q^\star}$ and $r_{\mathcal{G}^\star}$ are deterministic functions of $r_{\mathcal{Q}_q}$, $r_{\mathcal{G}}$, and $f_{\boldsymbol{W}}$.

- We consider the stationary random point measure

$$P^\star := \sum_{i=1}^{n} \delta\left[H_{\boldsymbol{W}_i}\left(\frac{R_i - r_{\mathcal{Q}_q^\star}(\boldsymbol{W}_i)}{r_{\mathcal{G}^\star}(\boldsymbol{W}_i)}\right)^{1/d} \boldsymbol{W}_i\right] \mathbb{1}_{R_i > r_{\mathcal{Q}_q^\star}(\boldsymbol{W}_i)}.$$

- We use an adapted version of the standard K-functions to assess if $P^\star$ is statistically distinguishable from a random point measure with constant intensity on $B_1(\boldsymbol{0})$.

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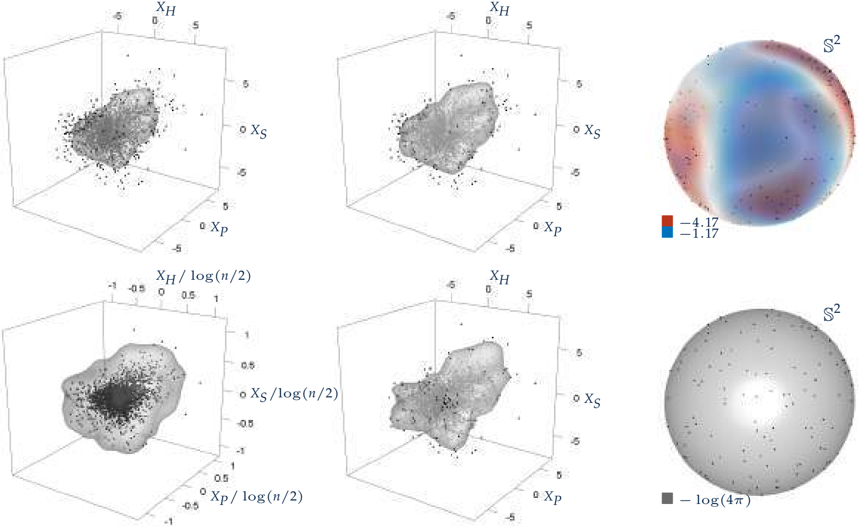
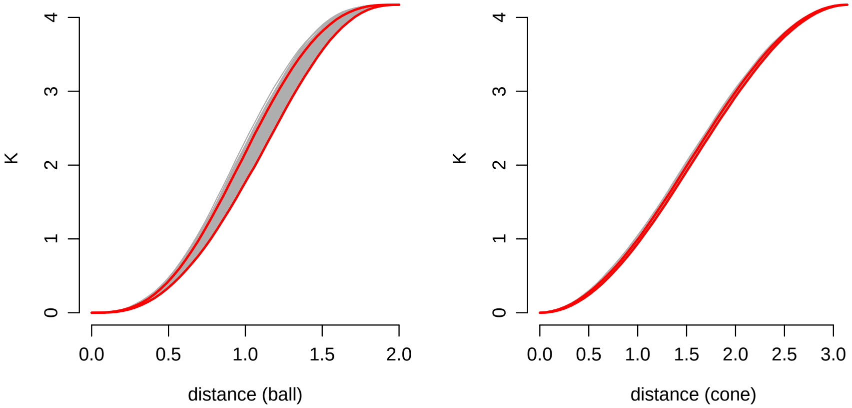


Figure from Papastathopoulos et al. (2023)



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